

MAS212 Linear Algebra I 2007-2008

Coursework 2 — Solutions

Exercise 1. The solution set of a homogeneous linear equation in three unknowns

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

can be interpreted geometrically as a plane in $\mathbb{R}^3$, provided that not all of the a_i 's are zero. More precisely, in this case the solution set is the plane of all 3-vectors which are orthogonal to the coefficient vector $(a_1, a_2, a_3)^T$. In particular, this plane contains the origin. If all a_i 's are zero, then the solution set is all of $\mathbb{R}^3$.

Thus, the solution set of a homogeneous 2×3 system can be interpreted geometrically as either the intersection of two planes through the origin (giving a line or a plane through the origin), or the intersection of a plane through the origin and $\mathbb{R}^3$ (giving that plane through the origin), or the intersection of $\mathbb{R}^3$ and $\mathbb{R}^3$ (which is, of course, $\mathbb{R}^3$). In all cases, the intersection contains infinitely many points in $\mathbb{R}^3$.

Exercise* 2. (a)

$$(i) \quad A + B = \begin{pmatrix} 1 & -2 \\ -4 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1+2 & -2+3 \\ -4-1 & 3+5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -5 & 8 \end{pmatrix}$$

$$(ii) \quad 2A - 3B = 2 \begin{pmatrix} 1 & -2 \\ -4 & 3 \end{pmatrix} - 3 \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -8 & 6 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ -3 & 15 \end{pmatrix} = \begin{pmatrix} -4 & -13 \\ -5 & -9 \end{pmatrix}$$

(b) Only the products AA , BA , CB , and CC exist. For BA we find

$$BA = \begin{pmatrix} 3 & 0 \\ 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 3 \cdot (-2) + 0 \cdot 1 & 3 \cdot (-1) + 0 \cdot (-3) \\ 1 \cdot (-2) + (-1) \cdot 1 & 1 \cdot (-1) + (-1) \cdot (-3) \\ 0 \cdot (-2) + 2 \cdot 1 & 0 \cdot (-1) + 2 \cdot (-3) \end{pmatrix} = \begin{pmatrix} -6 & -3 \\ -3 & 2 \\ 2 & -6 \end{pmatrix}.$$

Similarly, for the other products

$$A^2 = \begin{pmatrix} 3 & 5 \\ -5 & 8 \end{pmatrix}, \quad CB = \begin{pmatrix} 3 & 2 \\ -7 & -2 \\ -1 & 5 \end{pmatrix}, \quad C^2 = \begin{pmatrix} 1 & -1 & 3 \\ -9 & 4 & -3 \\ 3 & -4 & 4 \end{pmatrix}.$$

Exercise* 3. Observe that

$$(A + B)(A - B) = A^2 + BA - AB - B^2.$$

Thus $(A + B)(A - B) = A^2 - B^2$ if and only if $BA - AB = O$, that is, A and B commute.

Exercise 4. Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be upper triangular, i.e. $a_{ij} = b_{ij} = 0$ whenever $i > j$. First we show that $A + B$ is upper triangular. For the proof let $C = (c_{ij})_{n \times n}$ with $C = A + B$. Then $c_{ij} = a_{ij} + b_{ij}$ by the definition of matrix addition. We need to show that $c_{ij} = 0$ whenever $i > j$. To see this fix i and j with $i > j$. Then

$$c_{ij} = a_{ij} + b_{ij} = 0,$$

thus C is indeed upper triangular as claimed.

Next we show that AB is upper triangular. For the proof let $C = (c_{ij})_{n \times n}$ with $C = AB$. Then

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj},$$

by the definition of matrix multiplication. Fix i and j with $i > j$. We need to show that $c_{ij} = 0$. But since A and B are upper triangular we know that

$$\begin{aligned} a_{ik} &= 0 \quad \text{for } i > k \\ b_{kj} &= 0 \quad \text{for } k > j \end{aligned}$$

Thus

$$c_{ij} = \sum_{k=1}^j \underbrace{a_{ik}}_{=0} b_{kj} + \sum_{k=j+1}^n a_{ik} \underbrace{b_{kj}}_{=0} = 0,$$

which means that C is indeed upper triangular as claimed.

Exercise* 5. Let $A = (a_{ij})_{n \times n}$ and $D = (d_{ij})_{n \times n}$, where $d_{ii} = \alpha_i$ and $d_{ij} = 0$ whenever $i \neq j$.

(a) The (i, j) -entry of AD is

$$\sum_{k=1}^n a_{ik} d_{kj} = a_{ij} d_{jj} = \alpha_j a_{ij},$$

so the j -th column of AD is α_j times the j -th column of A .

(b) The (i, j) -entry of DA is

$$\sum_{k=1}^n d_{ik} a_{kj} = d_{ii} a_{ij} = \alpha_i a_{ij},$$

so the i -th column of DA is α_i times the i -th column of A .

Exercise* 6. Let $A = (a_{ij})$ and $B = (b_{ij})$.

(a) Clearly $(\alpha A)^T$ and $\alpha(A^T)$ have the same size, so we only need to check that they have the same entries. The (i, j) -entry of αA equals αa_{ij} , so the (i, j) -entry of $(\alpha A)^T$ equals αa_{ji} , which in turn is equal to the (i, j) -entry of $\alpha(A^T)$. Thus $(\alpha A)^T$ and $\alpha(A^T)$ have the same entries as claimed, and it follows that $(\alpha A)^T = \alpha(A^T)$.

(b) Again, since $(A + B)^T$ and $A^T + B^T$ have the same size we only need to check that they have the same entries. To see this, note that the (i, j) -entry of $A + B$ equals $a_{ij} + b_{ij}$, so the (i, j) -entry of $(A + B)^T$ equals $a_{ji} + b_{ji}$, which in turn is equal to the (i, j) -entry of $A^T + B^T$. Thus $(A + B)^T$ and $A^T + B^T$ have the same entries as claimed, and it follows that $(A + B)^T = A^T + B^T$.

Exercise* 7. Suppose that A is invertible. Then

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

Thus A^T is invertible with inverse $(A^{-1})^T$, that is

$$(A^T)^{-1} = (A^{-1})^T.$$